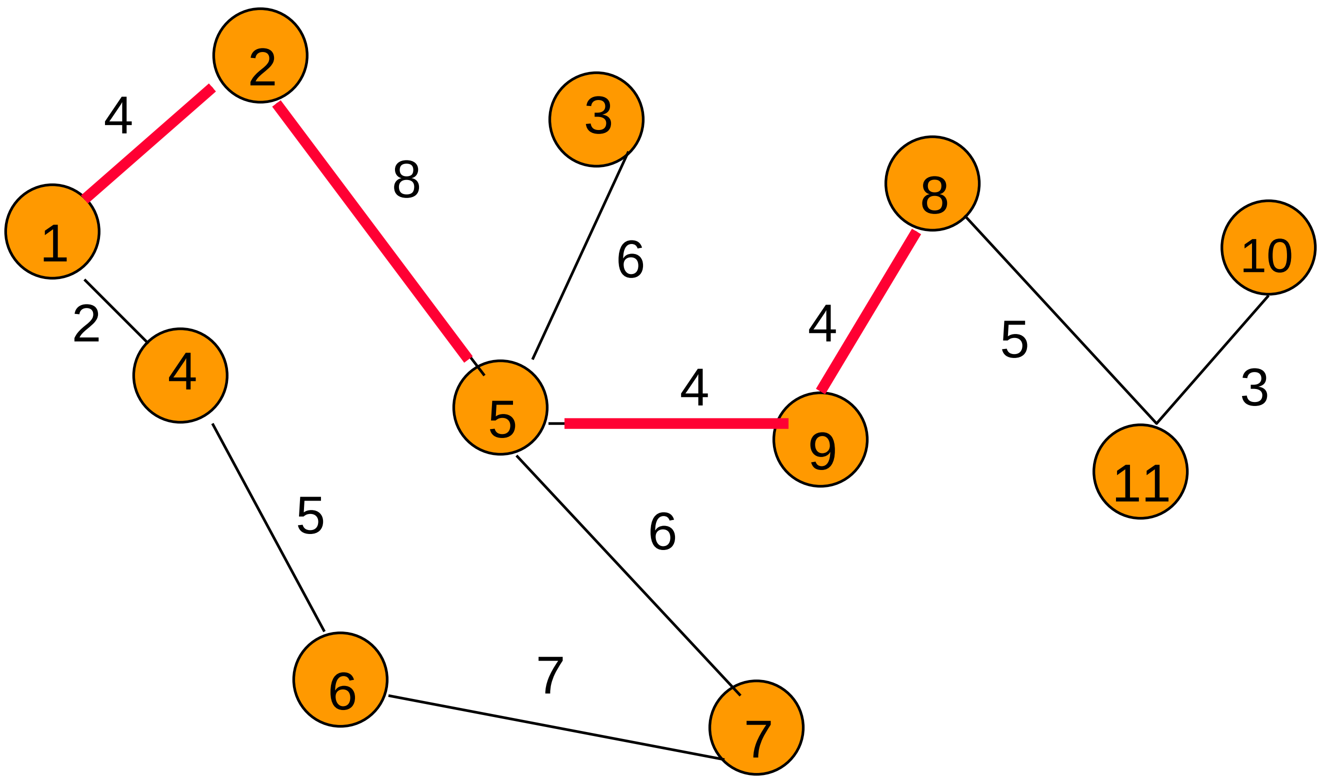
Chapter 9
Graph algorithms

Sample Graph Problems

- Path problems.
- Connectedness problems.
- Spanning tree problems.

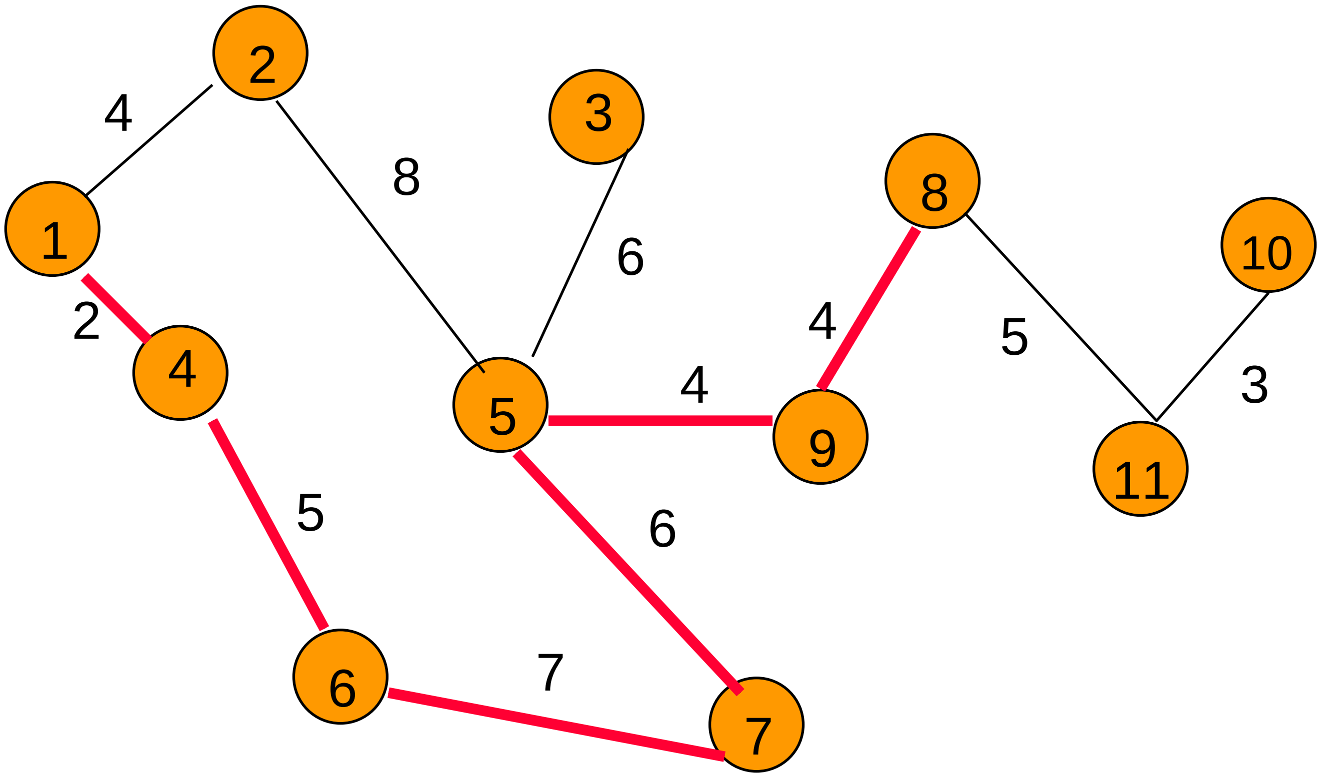
Path Finding

Path between 1 and 8.



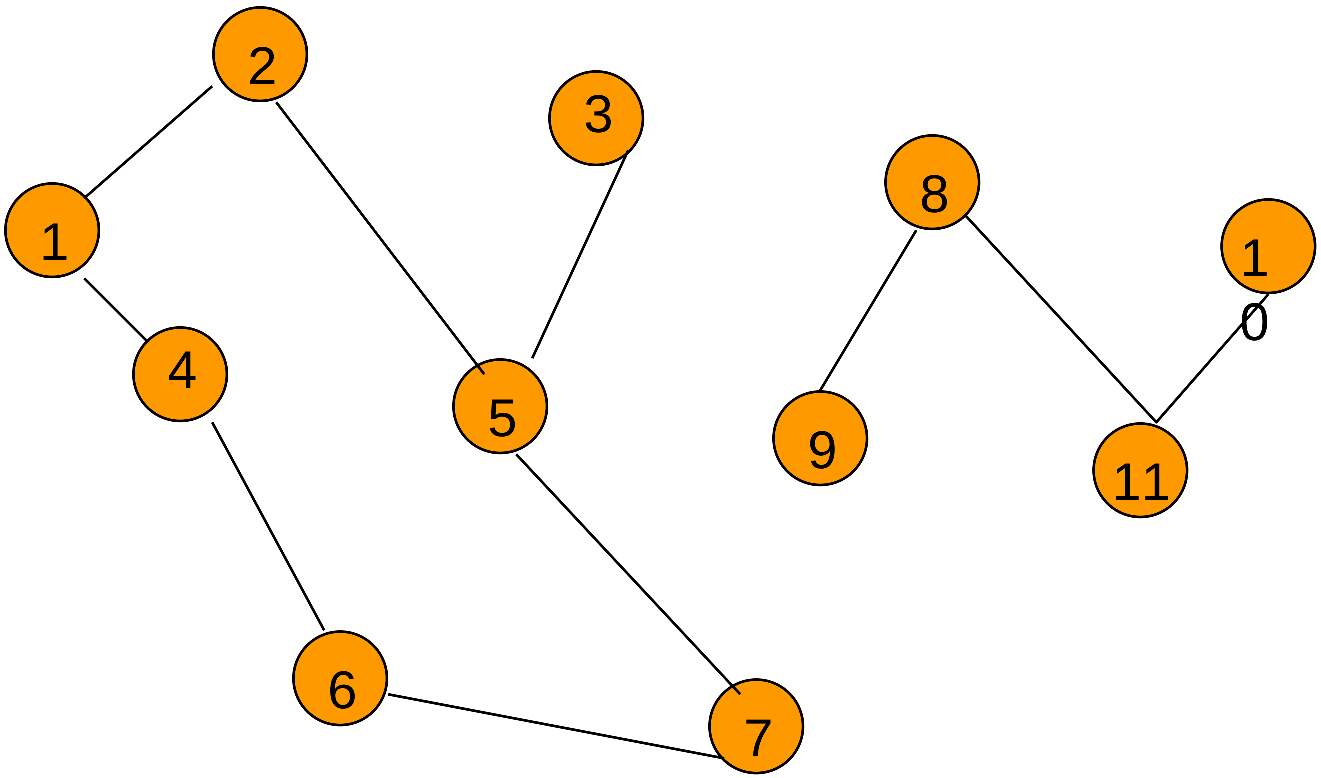
Path length is 20.

Another Path Between 1 and 8



Path length is 28.

Example Of No Path

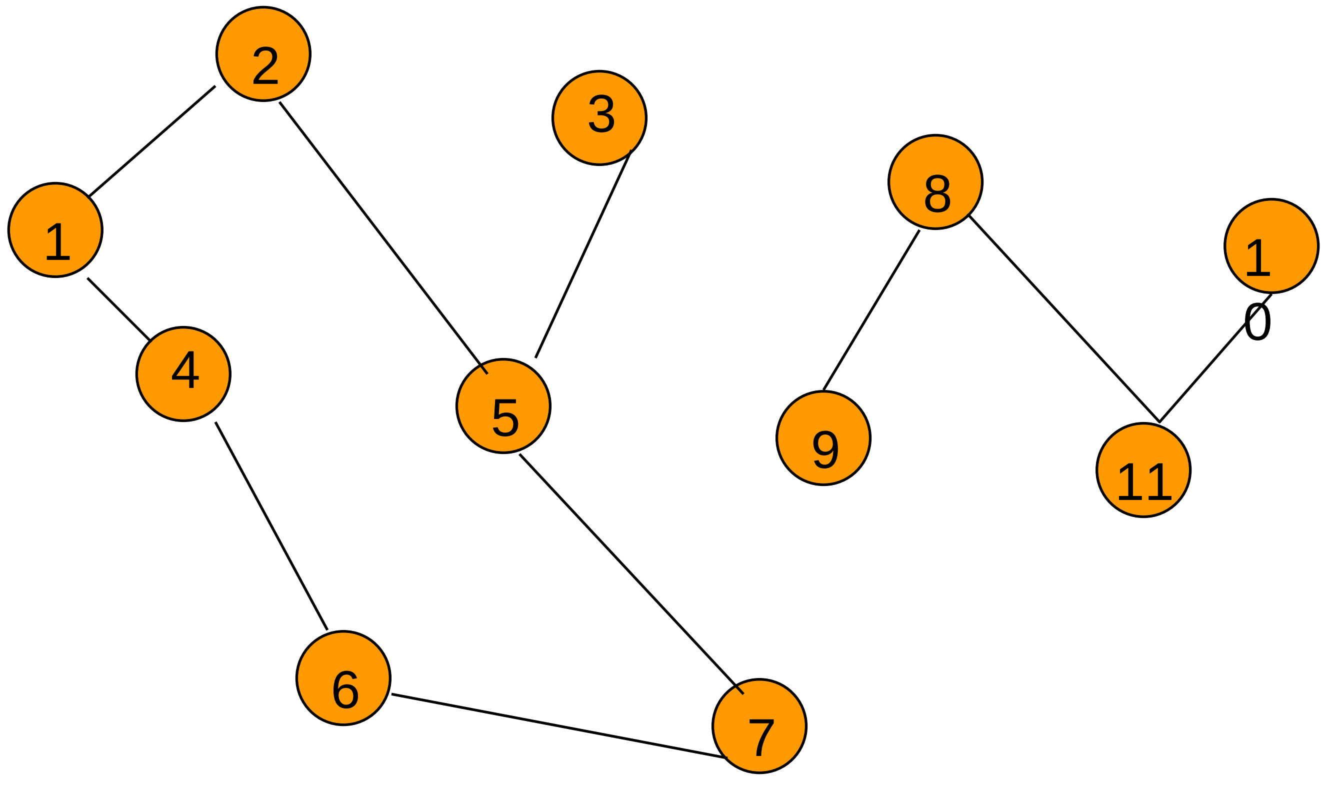


No path between 2 and 9.

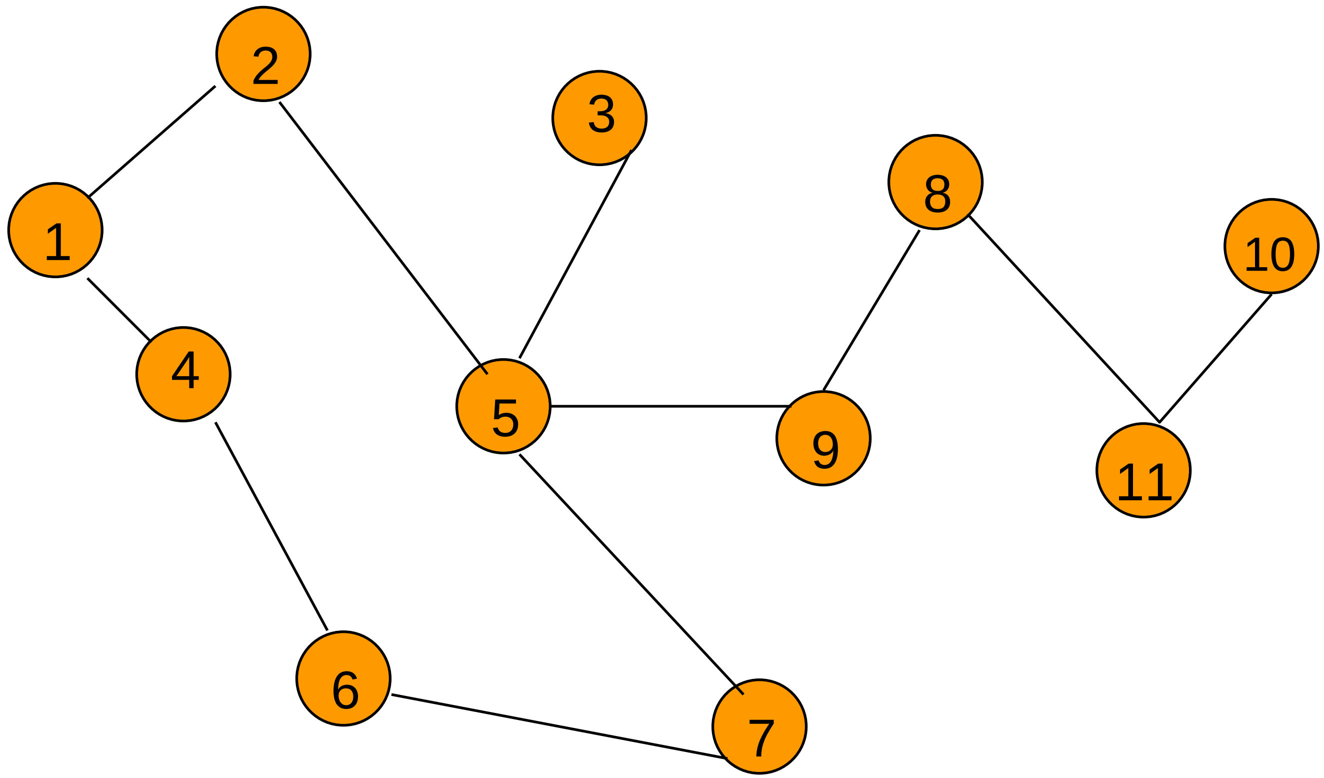
Connected Graph

- Undirected graph.
- There is a path between every pair of vertices.

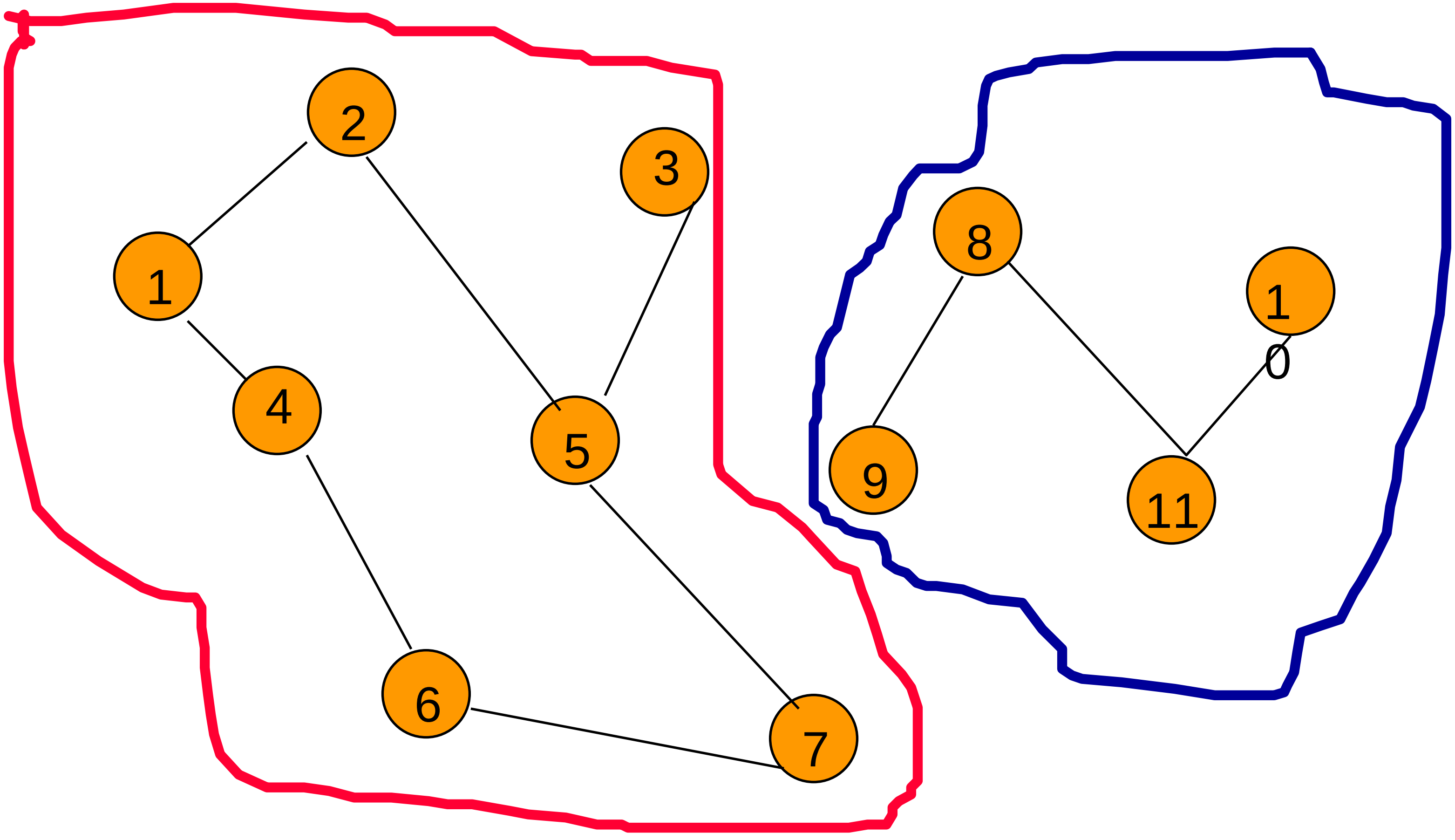
Example of a graph Not Connected



Connected Graph Example



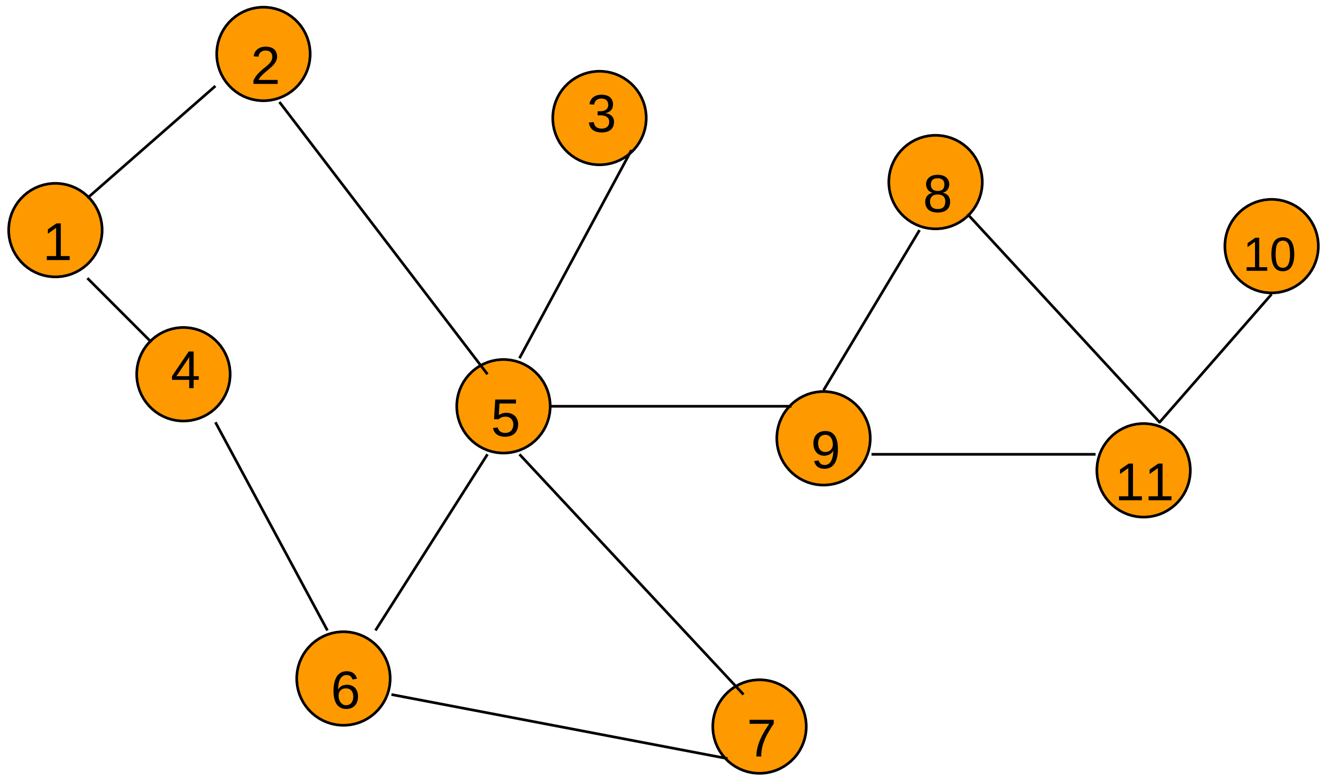
Connected Components



Connected Component

- A maximal subgraph that is connected.
  - Cannot add vertices and edges from original graph and retain connectedness.
- A connected graph has exactly 1 component.

Communication Network

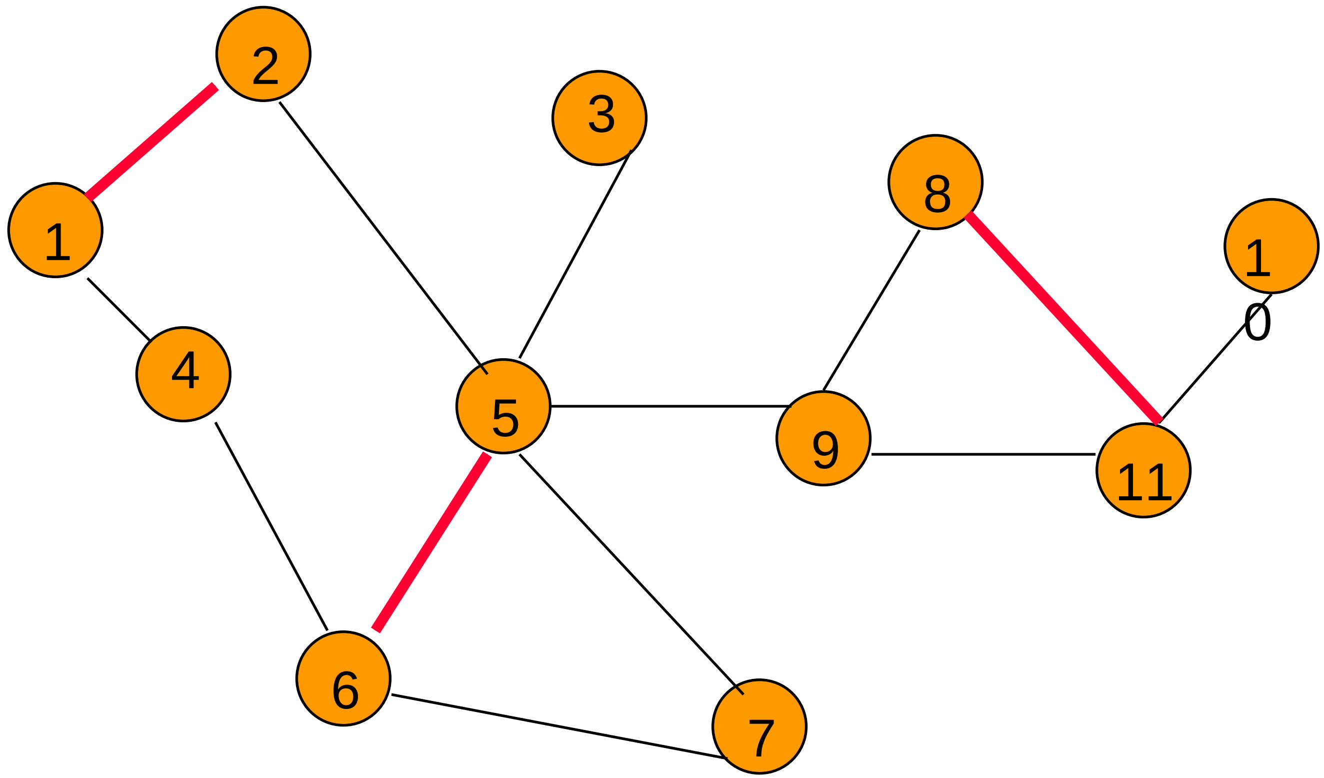


Each edge is a link that can be constructed (i.e., a feasible link).

Communication Network Problems

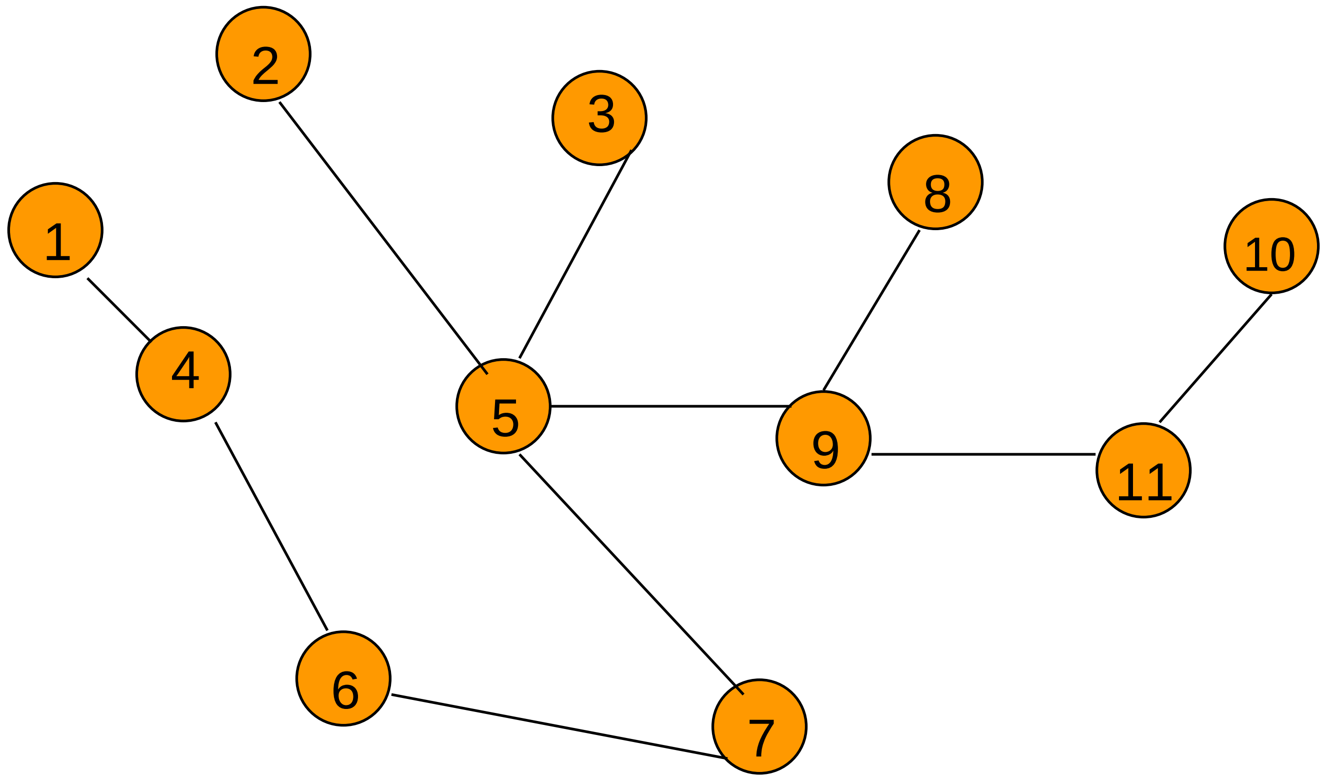
- Is the network connected?
  - Can we communicate between every pair of cities?
- Find the components.
- Want to construct smallest number of feasible links so that resulting network is connected.

Cycles And Connectedness



Removal of an edge that is on a cycle does not affect connectedness.

Cycles And Connectedness



Connected subgraph with all vertices and minimum number of edges has no cycles.

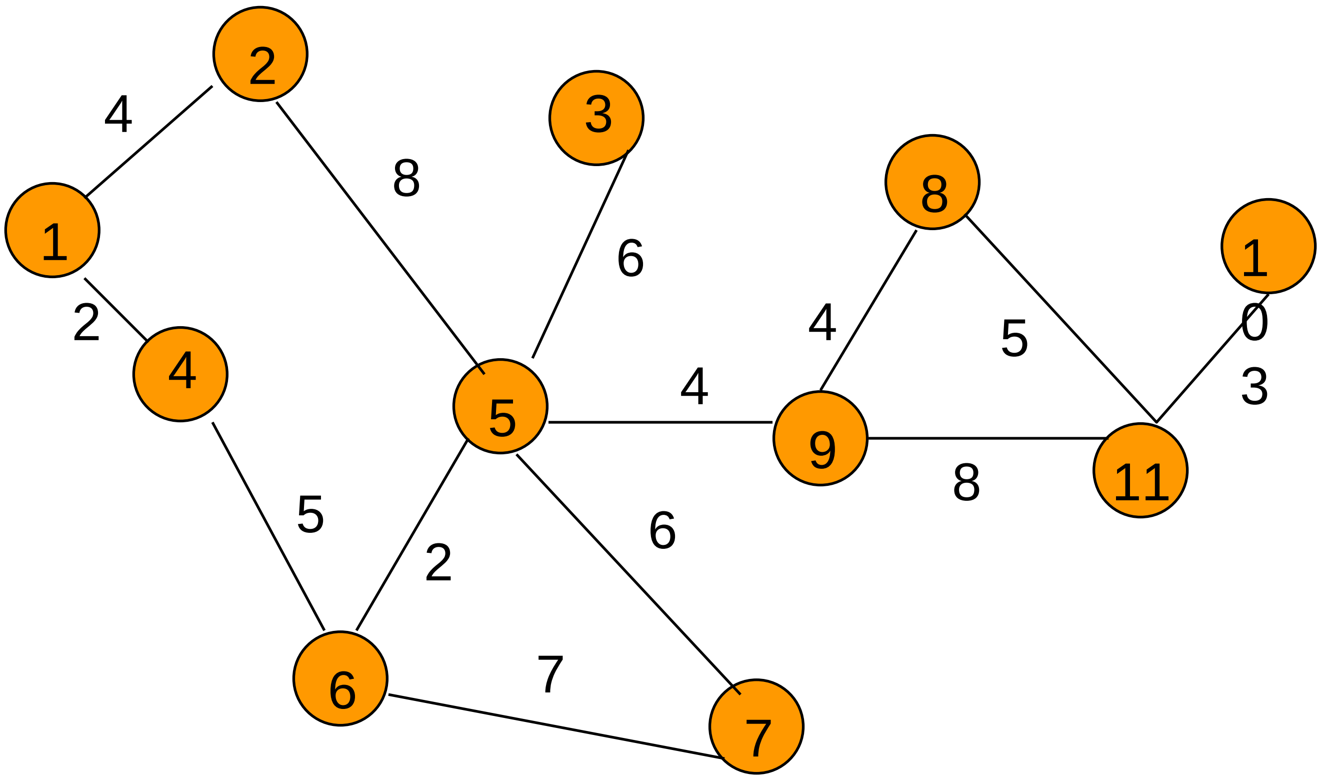
Tree

- Connected graph that has no cycles.
- n vertex connected graph with n-1 edges.
- A connected graph in which removal of any edge makes it unconnected.
- An cyclic graph in which addition of any edges introduces a cycle.

Spanning Tree

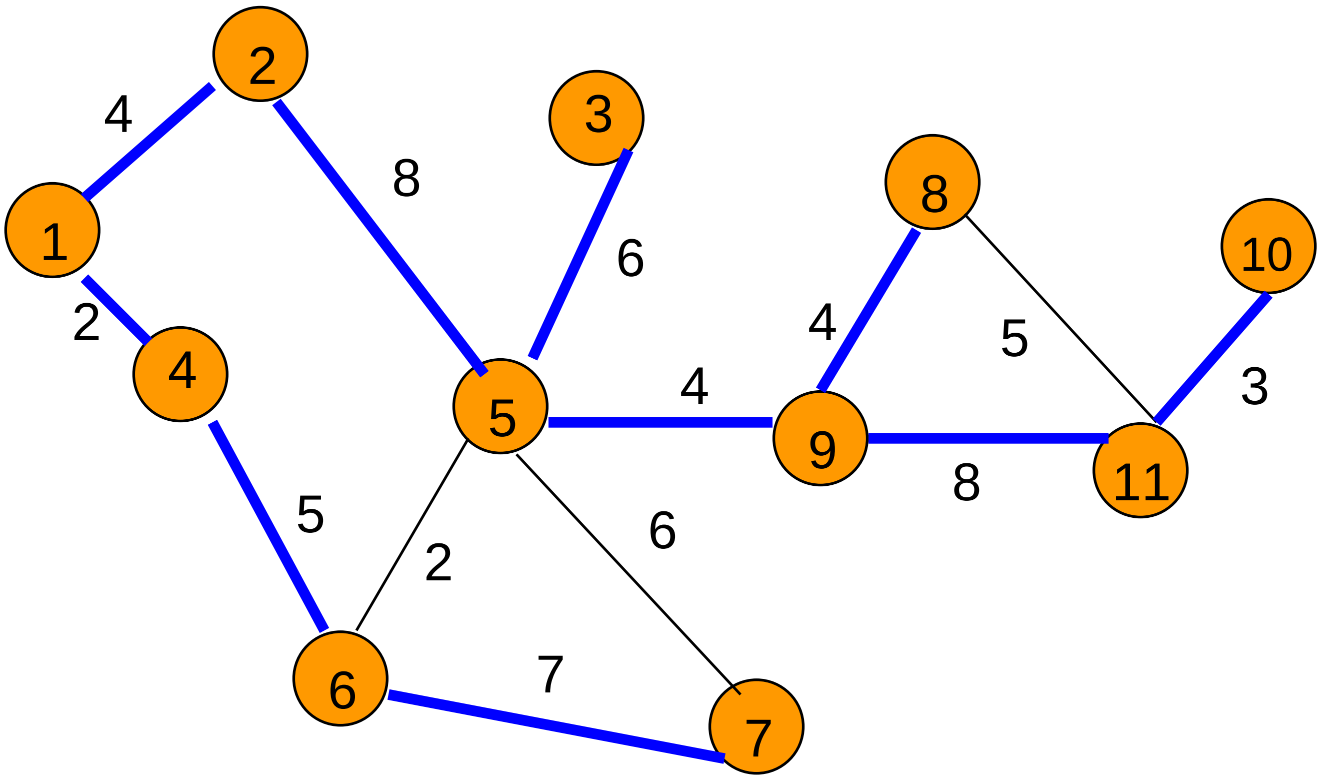
- Subgraph that includes all vertices of the original graph.
- Subgraph is a tree.
  - If original graph has n vertices, the spanning tree has n vertices and n-1 edges.

Minimum Cost Spanning Tree



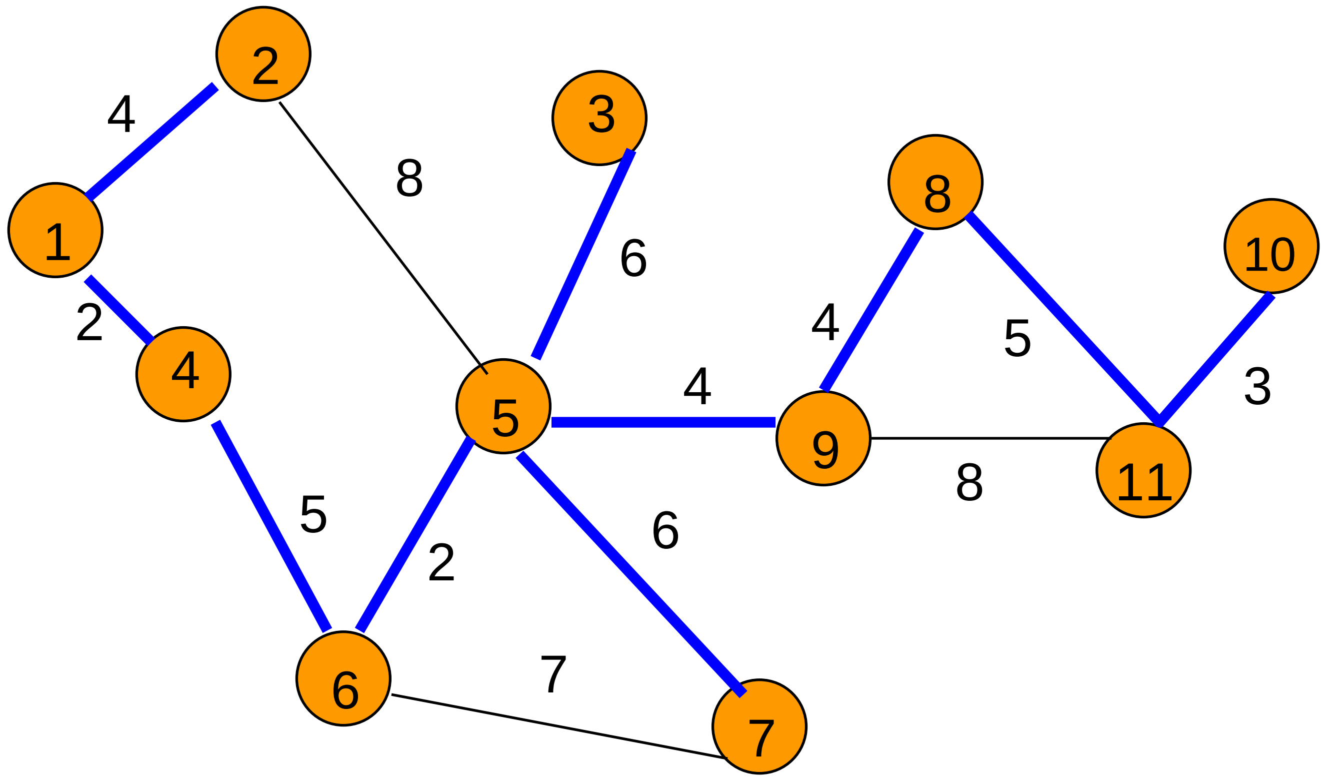
- Tree cost is sum of edge weights/costs.

A Spanning Tree



Spanning tree cost = 51.

Minimum Cost Spanning Tree



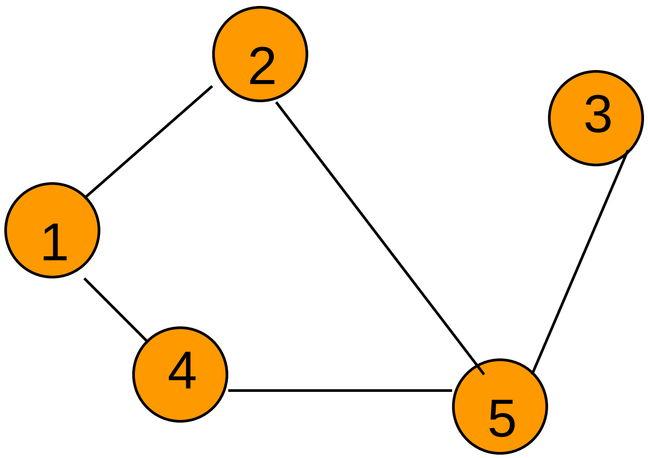
Spanning tree cost = 41.

Graph Representation

- Adjacency Matrix
- Adjacency Lists
  - Linked Adjacency Lists
  - Array Adjacency Lists

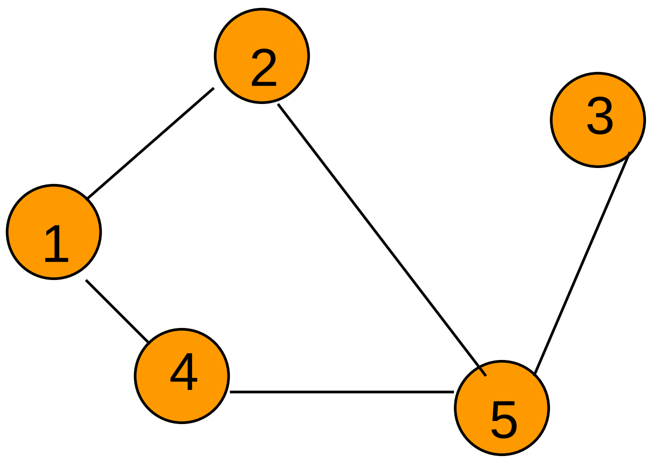
Adjacency Matrix

- 0/1 n x n matrix, where n = # of vertices
- A[i,j] = 1 iff (i,j) is an edge



	1	2	3	4	5
1	0	1	0	1	0
2	1	0	0	0	1
3	0	0	0	0	1
4	1	0	0	0	1
5	0	1	1	1	0

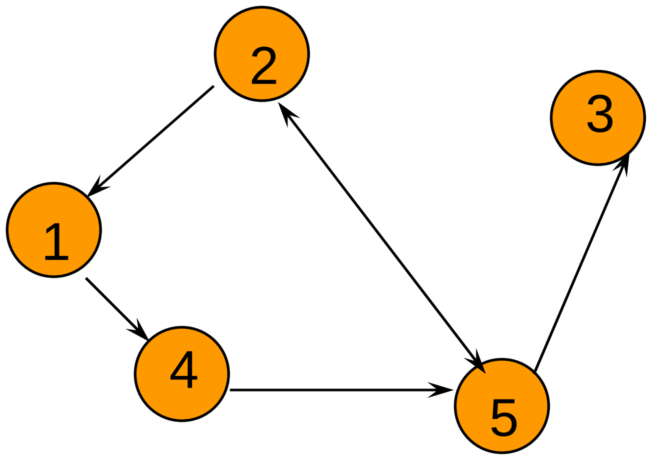
Adjacency Matrix Properties



	1	2	3	4	5
1	0	1	0	1	0
2	1	0	0	0	1
3	0	0	0	0	1
4	1	0	0	0	1
5	0	1	1	1	0

- Diagonal entries are zero.
- Adjacency matrix of an undirected graph is symmetric.
  - A(i,j) = A(j,i) for all i and j.

Adjacency Matrix (Digraph)



	1	2	3	4	5
1	0	0	0	1	0
2	1	0	0	0	1
3	0	0	0	0	0
4	0	0	0	0	1
5	0	1	1	0	0

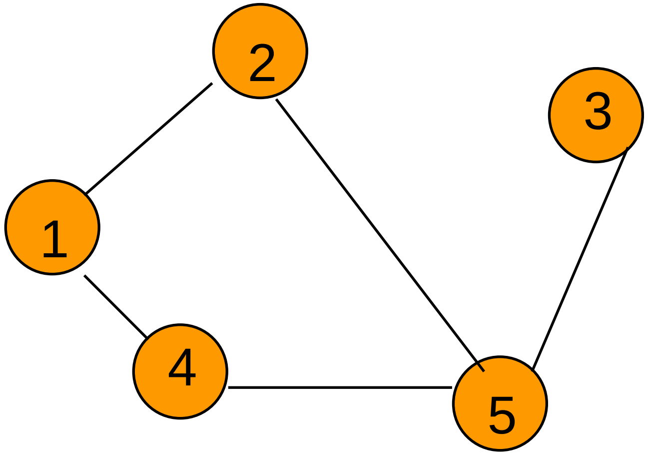
- Diagonal entries are zero.
- Adjacency matrix of a digraph need not be symmetric.

Adjacency Matrix

- n^2 bits of space
- For an undirected graph, may store only lower or upper triangle (exclude diagonal).
  - $(n-1)n/2$ bits
- $O(n)$ time to find vertex degree and/or vertices adjacent to a given vertex.
- $O(1)$ time to determine if there is an edge between two given vertices.

Adjacency Lists

- Adjacency list for vertex i is a linear list of vertices adjacent from vertex i.
- An array of n adjacency lists.



aList[1] = (2,4)

aList[2] = (1,5)

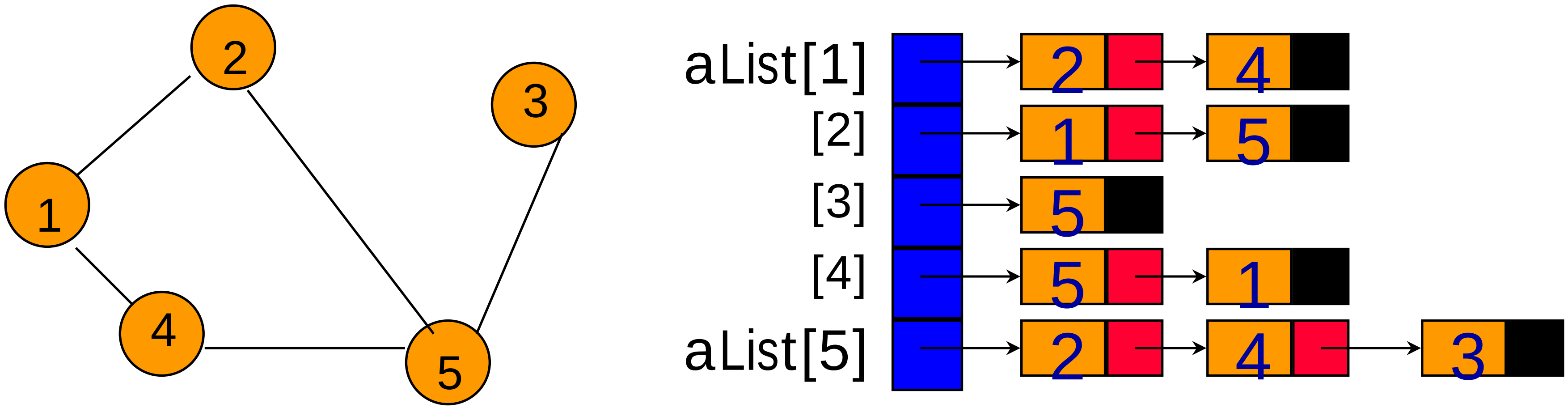
aList[3] = (5)

aList[4] = (5,1)

aList[5] = (2,4,3)

Linked Adjacency Lists

- Each adjacency list is a chain.



Array Length = n

# of chain nodes = 2e (undirected graph)

# of chain nodes = e (digraph)

Weighted Graphs

- Cost adjacency matrix.
  - C(i,j) = cost of edge (i,j)
- Adjacency lists => each list element is a pair (adjacent vertex, edge weight)

Single-source Shortest path problem

- directed, weighted graph is the input

- specified source s.

- want to compute the shortest path from s to all the vertices.

Example:

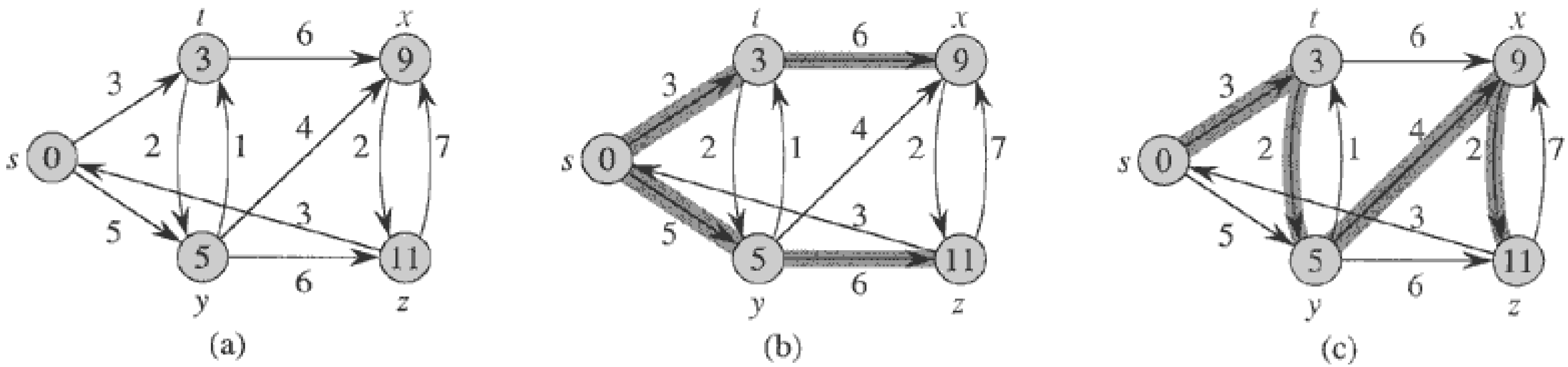


Figure 24.2 **(a)** A weighted, directed graph with shortest-path weights from source s. **(b)** The shaded edges form a shortest-paths tree rooted at the source s. **(c)** Another shortest-paths tree with the same root.

Dijkstra's algorithm:

- Works when there are no negative weight edges.

- takes time O(e log n) where e = number of edges, n = number of vertices.

- suppose $n \sim 10^5$, $e \sim 10^6$, then the number of computations $\sim 2 \times 10^6$

Data structures needed:

- Adjacency list rep. of graph
- a heap
- some additional structures (e.g. array)

INITIALIZE-SINGLE-SOURCE(G, s)

for each vertex $v \in V[G]$
    do $d[v] \leftarrow \infty$
        $\pi[v] \leftarrow$ NIL
$d[s] \leftarrow 0$

key operation: relaxation on edge

For each node v, the algorithm assigns a value d[v] which gets updated and will in the end become the length of the shortest path from s to v.

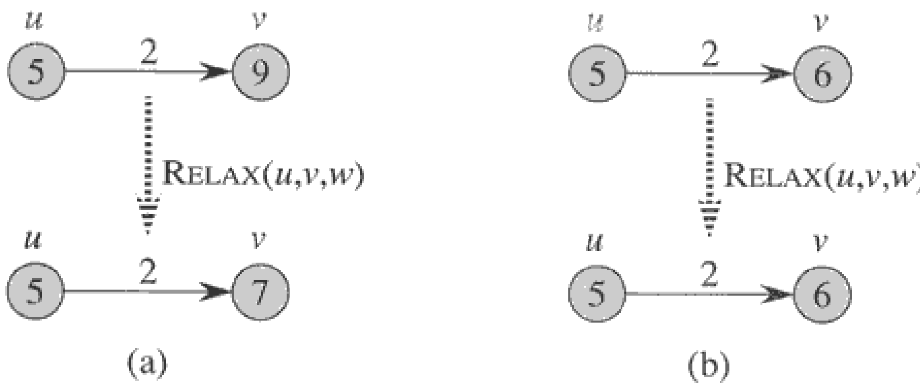


Figure 24.3 Relaxation of an edge (u, v) with weight $w(u, v) = 2$. The shortest-path estimate of each vertex is shown within the vertex. **(a)** Because $d[v] > d[u] + w(u, v)$ prior to relaxation, the value of $d[v]$ decreases. **(b)** Here, $d[v] \leq d[u] + w(u, v)$ before the relaxation step, and so $d[v]$ is unchanged by relaxation.

implementation of relaxation

RELAX(u, v, w)

if $d[v] > d[u] + w(u, v)$

  then $d[v] \leftarrow d[u] + w(u, v)$

    $\pi[v] \leftarrow u$

Dijkstra's algorithm

DIJKSTRA(G, w, s)

INITIALIZE-SINGLE-SOURCE(G, s)
$S \leftarrow \emptyset$
$Q \leftarrow V[G]$
while $Q \neq \emptyset$
  do $u \leftarrow$ EXTRACT-MIN(Q)
    $S \leftarrow S \cup \{u\}$
    for each vertex $v \in Adj[u]$
      do RELAX(u, v, w)

Dijkstra's algorithm – Example

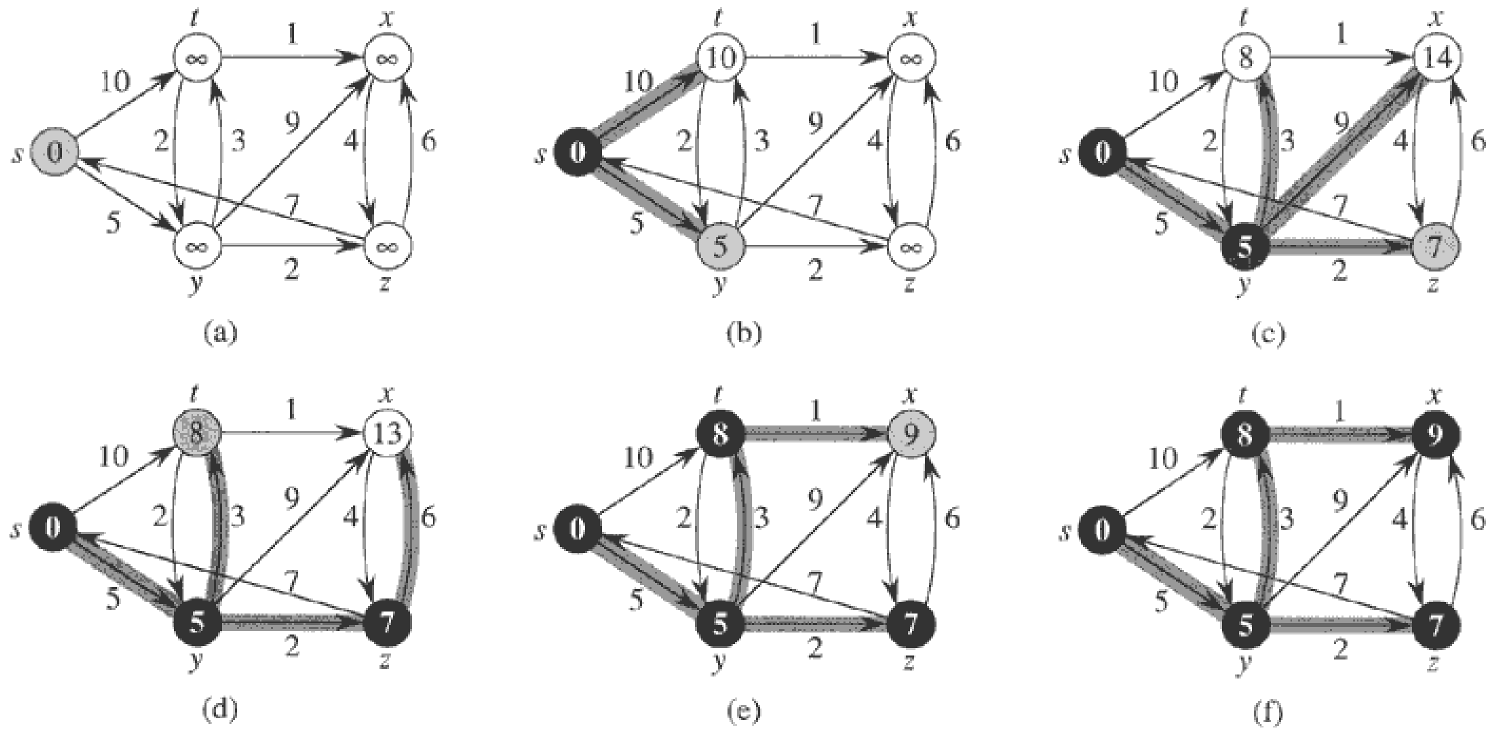


Figure 24.6 The execution of Dijkstra's algorithm. The source s is the leftmost vertex. The shortest-path estimates are shown within the vertices, and shaded edges indicate predecessor values. Black vertices are in the set S, and white vertices are in the min-priority queue $Q = V - S$. **(a)** The situation just before the first iteration of the **while** loop of lines 4–8. The shaded vertex has the minimum d value and is chosen as vertex u in line 5. **(b)–(f)** The situation after each successive iteration of the **while** loop. The shaded vertex in each part is chosen as vertex u in line 5 of the next iteration. The d and π values shown in part (f) are the final values.

Course summary